INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + c \Leftrightarrow \frac{d}{dx}\{F(x)+c\} = f(x)$, where c is called the **constant of integration.**

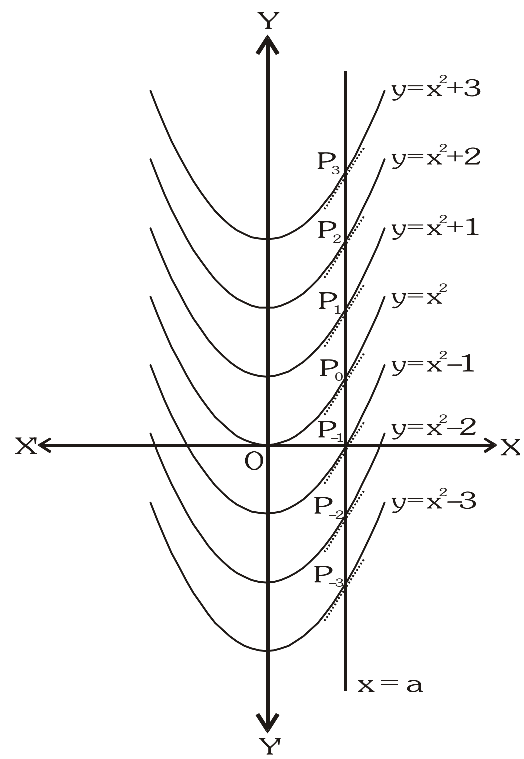
1. GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL :

$\int f(x)dx = F(x) + c = y(\text{say})$, represents a family of curves. The different values of c will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

Let f(x) = 2x. Then $\int f(x)dx = x^2 + c$. For different values of c, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + c$, where c is arbitrary constant, represents a family of integrals. By assigning different values to c, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

If the line x = a intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals 2a. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\,dx = x^2 + c = f(x) + c$ (say), **implies that the tangents to all the curves f(x) + c, c ∈ R, at the points of intersection of the curves by the line x = a, (a ∈ R) , are parallel.**



2. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\left| ax+b \right| + c$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a} + c,\ (a > 0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n\,|\sec(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \text{cosec}\,(ax+b).\cot(ax+b)dx = -\frac{1}{a}\text{cosec}\,(ax+b) + c$

(xii) $\int \sec\,(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

(xiii) $\int \sec x\, dx = \ell n|\sec x + \tan x| + c = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

(xiv) $\int \text{cosec}\, x\, dx = \ell n|\text{cosec}\, x - \cot x| + c = \ell n\left|\tan\frac{x}{2}\right| + c = -\ell n\ |\ \text{cosec}\, x + \cot x\ | + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ell n\left[x + \sqrt{x^2 + a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ell n\left[x + \sqrt{x^2 - a^2}\right] + c$

(xx) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 + a^2}\right) + c$

(xxiv) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 - a^2}\right) + c$

(xxv) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

(xxvi) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

3. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ and $\phi'(x)dx = dt$.

Hence $I = \int f(\phi(x))\phi'(x)dx$ reduces to $\int f(t)dt$.

(i) Fundamental deductions of method of substitution :

$\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n} dx$ put $f(x) = t$ & proceed.

Illustration 1 : Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Solution : $I = \int \frac{(1 - \sin^2 x)\cos x}{\sin x(1 + \sin x)} dx = \int \frac{1 - \sin x}{\sin x}\cos x\, dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\Rightarrow I = \int \frac{1-t}{t} dt = \ell n\ |\ t\ | - t + c = \ell n\ |\ \sin x\ | - \sin x + c$ Ans.

Illustration 2 : Evaluate $\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \frac{1}{x}\right)}$

Solution : The given integral can be written as

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Let $\left(x+\frac{1}{x}\right) = t$. Differentiating we get $\left(1-\frac{1}{x^2}\right)dx = dt$

Hence $I = \int \frac{dt}{(t^2+1)\tan^{-1} t}$

Now make one more substitution $\tan^{-1}t = u$. Then $\frac{dt}{t^2+1} = du$ and $I = \int \frac{du}{u} = \ell n\,|u| + c$

Returning to t, and then to x, we have

$I = \ell n\,|\tan^{-1} t| + c = \ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + c$ **Ans.**

Do yourself -1 :

(i) Evaluate : $\int \frac{x^2}{9+16x^6}dx$ **(ii)** Evaluate : $\int \cos^3 x\,dx$

(ii) Standard substitutions :

$\int \frac{dx}{a^2+x^2}$ or $\int \sqrt{a^2+x^2}\,dx$; put $x = a\tan\theta$ or $x = a\cot\theta$

$\int \frac{dx}{a^2-x^2}$ or $\int \sqrt{a^2-x^2}\,dx$; put $x = a\sin\theta$ or $x = a\cos\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2}\,dx$; put $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$

$\int \sqrt{\frac{a-x}{a+x}}\,dx$; put $x = a\cos 2\theta$

$\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

Illustration 3 : Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution : Put $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I=\int\frac{2(b-a)\sin\theta\cos\theta d\theta}{\left\{(a\cos^2\theta+b\sin^2\theta-a)(b-a\cos^2\theta-b\sin^2\theta)\right\}^{\frac{1}{2}}}$$

$$=\int\frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta}=\left(\frac{b-a}{b-a}\right)\int 2d\theta=2\theta+c=2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$$ **Ans.**

Illustration 4 : Evaluate $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$

Solution : Put $x=\cos^2\theta \Rightarrow dx=-2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow \quad I=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\frac{1}{\cos^2\theta}(-2\sin\theta\cos\theta)d\theta=-\int 2\tan\frac{\theta}{2}\tan\theta\, d\theta$$

$$=-4\int\frac{\sin^2(\theta/2)}{\cos\theta}d\theta=-2\int\frac{1-\cos\theta}{\cos\theta}d\theta=-2\ell n\,|\sec\theta-\tan\theta|+2\theta+c$$

$$=-2\ell n\left|\frac{1+\sqrt{1-x}}{x}\right|+2\cos^{-1}\sqrt{x}+c$$

Do yourself -2 :

(i) Evaluate : $\int\sqrt{\frac{x-3}{2-x}}dx$ (ii) Evaluate : $\int\frac{dx}{x\sqrt{x^2+4}}$

(b) **Integration by part :** $\int u.v\,dx=u\int v\,dx-\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions and are commonly designated as first & second function respectively.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

Illustration 5 : Evaluate : $\int\cos\sqrt{x}\,dx$

Solution : Consider $I=\int\cos\sqrt{x}dx$

Let $\sqrt{x}=t$ then $\frac{1}{2\sqrt{x}}dx=dt$

i.e. $dx=2\sqrt{x}dt$ or $dx=2t\,dt$

so $I=\int\cos t.2t dt$

taking t as first function, then integrate it by part

$$\Rightarrow \quad I=2\left[t\int\cos t dt-\int\left\{\frac{dt}{dt}\int\cos t dt\right\}dt\right]=2\left[t\sin t-\int 1.\sin t dt\right]=2[t\sin t+\cos t]+c$$

$$I=2\left[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right]+c$$ **Ans.**

Illustration 6 : Evaluate : $\int\frac{x}{1+\sin x}dx$

Solution : Let $I = \int \frac{x}{1+\sin x}dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$$= \int \frac{x(1-\sin x)}{1-\sin^2 x}dx = \int \frac{x(1-\sin x)}{\cos^2 x}dx = \int x\sec^2 x dx - \int x \sec x \tan x dx$$

$$= \left[x\int \sec^2 x dx - \int\left\{\frac{dx}{dx}\int \sec^2 x dx\right\}dx\right] - \left[x\int \sec x \tan x dx - \int\left\{\frac{dx}{dx}\int \sec x \tan x dx\right\}dx\right]$$

$$= \left[x\tan x - \int \tan x dx\right] - \left[x \sec x - \int \sec x dx\right]$$

$$= [x\tan x - \ell n|\sec x|] - [x\sec x - \ell n|\sec x + \tan x|] + c$$

$$= x(\tan x - \sec x) + \ell n\left|\frac{(\sec x + \tan x)}{\sec x}\right| + c = \frac{-x(1-\sin x)}{\cos x} + \ell n|1+\sin x| + c$$ **Ans.**

Do yourself -3 :

(i) Evaluate : $\int xe^x dx$ (ii) Evaluate : $\int x^3 \sin(x^2)dx$

Two classic integrands :

(i) $\int e^x[f(x) + f'(x)]dx = e^x.f(x) + c$

Illustration 7 : Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution : $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x \frac{(1-2x+x^2)}{(1+x^2)^2}dx = \int e^x\left(\frac{1}{(1+x^2)} - \frac{2x}{(1+x^2)^2}\right)dx = \frac{e^x}{1+x^2} + c$ **Ans.**

Illustration 8 : The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to -

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}$ (B) $\frac{e^x(1-x+x^2)}{(1+x^2)^{3/2}}$ (C) $\frac{e^x(1-x)}{(1+x^2)^{3/2}}$ (D) none of these

Solution : Let $I = \int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx = \int e^x\left(\frac{1}{(1+x^2)^{1/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right).dx$

$$= \int e^x\left(\frac{1}{(1+x^2)^{1/2}} - \frac{x}{(1+x^2)^{3/2}} + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$= \frac{e^x}{(1+x^2)^{1/2}} + \frac{xe^x}{(1+x^2)^{3/2}} + c = \frac{e^x\{1+x^2+x\}}{(1+x^2)^{3/2}} + c$$ **Ans. (D)**

Do yourself -4 :

(i) Evaluate : $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$ (ii) Evaluate : $\int xe^{x^2}\left(\sin x^2 + \cos x^2\right)dx$

(ii) $\int [f(x) + xf'(x)]dx = x f(x) + c$

Illustration 9 : Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$

Solution : $I = \int \frac{x+\sin x}{1+\cos x}dx = \int\left(\frac{x+\sin x}{2\cos^2\frac{x}{2}}\right)dx = \int\left(x\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx = x\tan\frac{x}{2} + c$ **Ans.**

> **Do yourself -5 :**
>
> (i) Evaluate : $\int\left(\tan(e^x)+xe^x\sec^2(e^x)\right)dx$ (ii) Evaluate : $\int(\ell nx+1)dx$

(c) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
4.	$\frac{f(x)}{(x-a)(x^2+bx+c)^2}$ where f(x) is a polynomial of degree less than 5.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}+\frac{Dx+E}{(x^2+bx+c)^2}$

Illustration 10 : Evaluate $\int\frac{x}{(x-2)(x+5)}dx$

Solution : $\frac{x}{(x-2)(x+5)} = \frac{A}{x-2}+\frac{B}{x+5}$

or $x = A(x+5) + B(x-2)$.

by comparing the coefficients, we get

$A = 2/7$ and $B = 5/7$ so that

$$\int\frac{x}{(x-2)(x+5)}dx = \frac{2}{7}\int\frac{dx}{x-2}+\frac{5}{7}\int\frac{dx}{x+5} = \frac{2}{7}\ell n|(x-2)|+\frac{5}{7}\ell n|(x+5)|+c$$ **Ans.**

Illustration 11 : Evaluate $\int \frac{x^4}{(x+2)(x^2+1)}dx$

Solution :

$$\frac{x^4}{(x+2)(x^2+1)} = (x-2) + \frac{3x^2+4}{(x+2)(x^2+1)}$$

Now, $\frac{3x^2+4}{(x+2)(x^2+1)} = \frac{16}{5(x+2)} + \frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

So, $\frac{x^4}{(x+2)(x^2+1)} = x-2+\frac{16}{5(x+2)} + \frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

Now, $\int\left((x-2)+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}\right)dx$

$= \frac{x^2}{2} - 2x + \frac{2}{5}\tan^{-1}x + \frac{16}{5}\ell n|x+2| - \frac{1}{10}\ell n(x^2+1)+c$ **Ans.**

Do yourself - 6 :

(i) Evaluate : $\int \frac{3x+2}{(x+1)(x+3)}dx$ (ii) Evaluate : $\int \frac{x^2-1}{(x+1)(x+2)}dx$

(ii) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(iii) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express px + q = ℓ (differential coefficient of denominator) + m.

Illustration 12 : Evaluate $\int \frac{dx}{2x^2+x-1}$

Solution :

$$I = \int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2-9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2-(3/4)^2}$$

$$= \frac{1}{2}.\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right|+c \qquad \left\{\text{using}, \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right\}$$

$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right|+c = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+c$ **Ans.**

Illustration 13 : Evaluate $\int \frac{3x+2}{4x^2+4x+5}dx$

Solution : Express $3x + 2 = \ell(\text{d.c. of } 4x^2 + 4x + 5) + m$

or, $3x + 2 = \ell(8x + 4) + m$

Comparing the coefficients, we get

$8\ell = 3$ and $4\ell + m = 2 \Rightarrow \ell = 3/8$ and $m = 2 - 4\ell = 1/2$

$$\Rightarrow I = \frac{3}{8}\int\frac{8x+4}{4x^2+4x+5}dx + \frac{1}{2}\int\frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\int\frac{dx}{x^2+x+\frac{5}{4}} = \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right) + c \quad \textbf{Ans.}$$

Do yourself -7 :

(i) Evaluate : $\int\frac{dx}{x^2+x+1}$ **(ii)** Evaluate : $\int\frac{5x+4}{\sqrt{x^2+3x+2}}dx$

(iv) Integrals of the form $\int\frac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int\frac{x^2-1}{x^4+Kx^2+1}dx$ where K is any constant.

Divide N^r & D^r by x^2 & proceed.

Note : Sometimes it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions e.g.

* $\int\frac{2x^2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx + \int\frac{x^2-1}{x^4+1}dx$ * $\int\frac{2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx - \int\frac{x^2-1}{x^4+1}dx$

These integrals can be called as **Algebric Twins**.

Illustration 14 : Evaluate : $\int\frac{4}{\sin^4 x+\cos^4 x}dx$

Solution : $I = 4\int\frac{1}{\sin^4 x+\cos^4 x}dx = 4\int\frac{\sin^2 x+\cos^2 x}{\sin^4 x+\cos^4 x}dx$

$$= 4\int\frac{(\tan^2 x+1)\cos^2 x}{(\tan^4 x+1)\cos^4 x}dx = 4\int\frac{(\tan^2 x+1)\sec^2 x}{(\tan^4 x+1)}dx$$

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = 4\int\frac{1+t^2}{1+t^4}dt = 4\int\frac{1/t^2+1}{t^2+1/t^2}dt$$

Now, put $t - 1/t = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$$\Rightarrow I = 4\int\frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} = 2\sqrt{2}\tan^{-1}\frac{t-1/t}{\sqrt{2}} = 2\sqrt{2}\tan^{-1}\left(\frac{\tan x-1/\tan x}{\sqrt{2}}\right)+c \quad \textbf{Ans.}$$

Illustration 15 : Evaluate : $\int\frac{1}{x^4+5x^2+1}dx$

Solution : $I = \frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx + \frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx = \frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$$

{dividing N^r and D^r by x^2}

$$= \frac{1}{2}\int \frac{(1+1/x^2)}{(x-1/x)^2+7}dx - \frac{1}{2}\int \frac{(1-1/x^2)dx}{(x+1/x)^2+3} = \frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{7}\right)^2} - \frac{1}{2}\int \frac{du}{u^2+\left(\sqrt{3}\right)^2}$$

where $t = x - \frac{1}{x}$ and $u = x + \frac{1}{x}$

$$I = \frac{1}{2}.\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right) - \frac{1}{2}.\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right) + c$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right] + c$$ **Ans.**

Do yourself -8 :

(i) Evaluate : $\int \frac{x^2+1}{x^4-x^2+1}dx$ (ii) Evaluate : $\int \frac{1}{1+x^4}dx$

(d) Manipulating integrands :

(i) $\int \frac{dx}{x(x^n+1)}$, $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$, $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t^n$.

Illustration 16 : Evaluate : $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

Solution : Let $I = \int \frac{dx}{x^n(1+x^n)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$

Put $1 + \frac{1}{x^n} = t^n$, then $\frac{1}{x^{n+1}}dx = -t^{n-1}dt$

$$I = -\int \frac{t^{n-1}dt}{t} = -\int t^{n-2}dt = -\frac{t^{n-1}}{n-1} + c = \frac{-1}{n-1}\left(1+\frac{1}{x^n}\right)^{\frac{n-1}{n}} + c$$ **Ans.**

Do yourself -9 :

(i) Evaluate : $\int \frac{dx}{x(x^2+1)}$ (ii) Evaluate : $\int \frac{dx}{x^2(x^3+1)^{2/3}}$ (iii) Evaluate : $\int \frac{dx}{x^3(x^3+1)^{1/3}}$

(e) Integration of trigonometric functions :

(i) $\int \frac{dx}{a+b\sin^2 x}$ OR $\int \frac{dx}{a+b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Divide N^r & D^r by $\cos^2 x$ & put $\tan x = t$.

Illustration 17 : Evaluate : $\int \frac{dx}{2+\sin^2 x}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$I = \int \frac{\sec^2 x dx}{2\sec^2 x + \tan^2 x} = \int \frac{\sec^2 x dx}{2+3\tan^2 x}$$

Let $\sqrt{3}\tan x = t$ $\therefore$ $\sqrt{3}\sec^2 x\, dx = dt$

So $I = \frac{1}{\sqrt{3}}\int \frac{dt}{2+t^2} = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + c$ **Ans.**

Illustration 18 : Evaluate : $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$\therefore \quad I = \int \frac{\sec^2 x dx}{(2\tan x + 3)^2}$$

Let $2\tan x + 3 = t$, $\therefore$ $2\sec^2 x dx = dt$

$I = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + c = -\frac{1}{2(2\tan x + 3)} + c$ **Ans.**

Do yourself -10 :

(i) Evaluate : $\int \frac{dx}{1+4\sin^2 x}$ (ii) Evaluate : $\int \frac{dx}{3\sin^2 x + \sin x\cos x + 1}$

(ii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles & put $\tan\frac{x}{2} = t$

In this case $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, $x = 2\tan^{-1}t$; $dx = \frac{2dt}{1+t^2}$

Illustration 19 : Evaluate : $\int \frac{dx}{3\sin x + 4\cos x}$

Solution : $I = \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{dx}{3\left\{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\} + 4\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}} = \int \frac{\sec^2\frac{x}{2}dx}{4+6\tan\frac{x}{2} - 4\tan^2\frac{x}{2}}$

let $\tan\frac{x}{2} = t$, $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

so $I = \int \frac{2dt}{4+6t-4t^2} = \frac{1}{2}\int \frac{dt}{1-\left(t^2 - \frac{3}{2}t\right)} = \frac{1}{2}\int \frac{dt}{\frac{25}{16} - \left(t-\frac{3}{4}\right)^2}$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{5}{4}\right)}\ell n\left|\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right|+c \;=\frac{1}{5}\ell n\left|\frac{1+2\tan\frac{x}{2}}{4-2\tan\frac{x}{2}}\right|+c$$ **Ans.**

Do yourself -11 :

(i) Evaluate : $\int \frac{dx}{3+\sin x}$ (ii) Evaluate : $\int \frac{dx}{1+4\sin x+3\cos x}$

(iii) $\int \frac{a\cos x+b\sin x+c}{p\cos x+q\sin x+r}dx$

Express Numerator (N^r) = $\ell(D^r)$ + $m\frac{d}{dx}(D^r)$ + n & proceed.

Illustration 20 : Evaluate : $\int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}d\theta$

Solution : Write the Numerator = ℓ(denominator) + m(d.c. of denominator) + n

$\Rightarrow 2 + 3\cos\theta = \ell(\sin\theta+2\cos\theta+3) + m(\cos\theta - 2\sin\theta) + n.$

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant terms,

we get $3\ell + n = 2$, $2\ell + m = 3$, $\ell - 2m = 0 \Rightarrow \ell = 6/5$, $m = 3/5$ and $n = -8/5$

Hence $I = \int \frac{6}{5}d\theta + \frac{3}{5}\int \frac{\cos\theta-2\sin\theta}{\sin\theta+2\cos\theta+3}d\theta - \frac{8}{5}\int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

$= \frac{6}{5}\theta + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}I_3$ where $I_3 = \int \frac{d\theta}{\sin\theta+2\cos\theta+3}$

In I_3, put $\tan\frac{\theta}{2} = t \Rightarrow \sec^2\frac{\theta}{2}d\theta = 2dt$

$I_3 = 2\int \frac{dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = 2.\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$

Hence $I = \frac{6\theta}{5} + \frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)+c$ **Ans.**

Do yourself -12 :

(i) Evaluate : $\int \frac{\sin x}{\sin x+\cos x}dx$ (ii) Evaluate $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

(iv) $\int \sin^m x\cos^n x\,dx$

Case-I : When m & n $\in$ natural numbers.

* If one of them is odd, then substitute for the term of even power.
* If both are odd, substitute either of the term.
* If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

Case-II : m + n is a negative even integer.

* In this case the best substitution is tanx = t.

Illustration 21 : Evaluate $\int \sin^3 x \cos^5 x\, dx$

Solution : Put $\cos x = t$; $-\sin x\, dx = dt$.

so that $I = -\int (1-t^2).t^5 dt$

$$= \int (t^7 - t^5)dt = \frac{t^8}{8} - \frac{t^6}{6} = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c$$

Alternate :

Put $\sin x = t$; $\cos x\, dx = dt$

so that $I = \int t^3 (1-t^2)^2 dt = \int \left(t^3 - 2t^5 + t^7\right) dt$

$$= \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + c$$

Note : This problem can also be handled by successive reduction or by trignometric identities.

Illustration 22 : Evaluate $\int \sin^2 x \cos^4 x\, dx$

Solution : $\int \sin^2 x \cos^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cos 2x + 1}{2}\right)^2 dx = \int \frac{1}{8}\left(1-\cos 2x\right)\left(\cos^2 2x + 2\cos 2x + 1\right) dx$

$$= \frac{1}{8}\int \left(\cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(-\cos^3 2x - \cos^2 2x + \cos 2x + 1\right) dx = -\frac{1}{8}\int \left(\frac{\cos 6x + 3\cos 2x}{4} + \frac{1+\cos 4x}{2} - \cos 2x - 1\right) dx$$

$$= -\frac{1}{32}\left[\frac{\sin 6x}{6} + \frac{3\sin 2x}{2}\right] - \frac{1}{16}x - \frac{\sin 4x}{64} + \frac{\sin 2x}{16} + \frac{x}{8} + c$$

$$= -\frac{\sin 6x}{192} - \frac{\sin 4x}{64} + \frac{1}{64}\sin 2x + \frac{x}{16} + c$$

Illustration 23 : Evaluate $\int \frac{\sqrt{\sin x}}{\cos^{9/2} x} dx$

Solution : Let $I = \int \frac{\sin^{1/2} x}{\cos^{9/2} x} dx = \int \frac{dx}{\sin^{-1/2} x \cos^{9/2} x}$

Here $m + n = \frac{1}{2} - \frac{9}{2} = -4$ (negative even integer).

Divide Numerator & Denominator by $\cos^4 x$.

$I = \int \sqrt{\tan x} \sec^4 x\, dx = \int \sqrt{\tan x}(1+\tan^2 x)\sec^2 x dx$

$= \int \sqrt{t}(1+t^2)dt$ (using $\tan x = t$)

$$= \frac{2}{3}t^{3/2} + \frac{2}{7}t^{7/2} + c = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + c$$

Do yourself -13 :

(i) Evaluate : $\int \frac{\sin^2 x}{\cos^4 x} dx$ **(ii)** Evaluate : $\int \frac{\sqrt{\sin x}\, dx}{\cos^{5/2} x}$ **(iii)** Evaluate : $\int \sin^2 x \cos^5 x\, dx$

(f) Integration of Irrational functions :

(i) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

(ii) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x=\frac{1}{t}$

Illustration 24 : Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$

Solution : Let, $I=\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$ Put $x+1=t^2 \Rightarrow dx=2tdt$

$$\therefore \quad I=\int \frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}.(2t)dt = 2\int \frac{t^2+1}{t^4+t^2+1}dt = 2\int \frac{1+1/t^2}{t^2+1+1/t^2}dt$$

$$=2\int \frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}.dt = 2\int \frac{du}{u^2+(\sqrt{3})^2} \qquad \left\{\text{where } u=t-\frac{1}{t}\right\}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+c \qquad \textbf{Ans.}$$

Illustration 25 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Solution : Let, $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$ put $x-1=\frac{1}{t} \Rightarrow dx=-1/t^2\,dt$

$$I=\int \frac{-1/t^2dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}} = -\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+1/12}} = -\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+c$$

$$=-\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+c \qquad \textbf{Ans.}$$

Illustration 26 : Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Let, $I=\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x=\frac{1}{t}$, So that $dx=\frac{-1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2dt}{(1+1/t^2)\sqrt{1-1/t^2}} = -\int \frac{tdt}{(t^2+1)\sqrt{t^2-1}}$$

again let, $t^2=u$. So that $2t\,dt=du$.

$= \dfrac{-1}{2}\displaystyle\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\displaystyle\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear so that we put $u - 1 = z^2$ so that $du = 2z\,dz$

$$\therefore \quad I = -\frac{1}{2}\int \frac{2z\,dz}{(z^2+1+1)\sqrt{z^2}} = -\int \frac{dz}{(z^2+2)}$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right) + c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + c \qquad \textbf{Ans.}$$

Do yourself -14 :

(i) Evaluate $\displaystyle\int \frac{x}{(x-3)\sqrt{x+1}}dx$ (ii) Evaluate : $\displaystyle\int \frac{dx}{x^2\sqrt{1+x^2}}$

Miscelleneous Illustrations :

Illustration 27 : Evaluate $\displaystyle\int \frac{\cos^4 x\,dx}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}$

Solution :

$$I = \int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x+\cos^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cos^4 x}{\sin^6 x\{1+\cot^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cot^4 x\,\mathrm{cosec}^2 x\,dx}{(1+\cot^5 x)^{3/5}}$$

Put $1 + \cot^5 x = t$

$5\cot^4 x\ \mathrm{cosec}^2 x\,dx = -dt$

$$= -\frac{1}{5}\int \frac{dt}{t^{3/5}} = -\frac{1}{2}t^{2/5} + c = -\frac{1}{2}(1+\cot^5 x)^{2/5} + c \qquad \textbf{Ans.}$$

Illustration 28 : $\displaystyle\int \frac{dx}{\cos^6 x + \sin^6 x}$ is equal to -

(A) $\ell n|\tan x - \cot x| + c$ (B) $\ell n|\cot x - \tan x| + c$

(C) $\tan^{-1}(\tan x - \cot x) + c$ (D) $\tan^{-1}(-2\cot 2x) + c$

Solution :

$$\text{Let } I = \int \frac{dx}{\cos^6 x+\sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x}dx = \int \frac{(1+\tan^2 x)^2\sec^2 x\,dx}{1+\tan^6 x}$$

If $\tan x = p$, then $\sec^2 x\,dx = dp$

$$\Rightarrow \quad I = \int \frac{(1+p^2)^2\,dp}{1+p^6} = \int \frac{(1+p^2)}{p^4-p^2+1}dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2+\frac{1}{p^2}-1\right)}dp$$

$$= \int \frac{dk}{k^2+1} = \tan^{-1}(k) + c \qquad \left(\text{where } p - \frac{1}{p} = k, \left(1+\frac{1}{p^2}\right)dp = dk\right)$$

$$= \tan^{-1}\left(p - \frac{1}{p}\right) + c = \tan^{-1}(\tan x - \cot x) + c = \tan^{-1}(-2\cot 2x) + c \qquad \textbf{Ans. (C,D)}$$

Illustration 29 : Evaluate : $\displaystyle\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}dx$

Solution :

$$I = \int \frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x}dx = \int \frac{(4\sin x - 1)\cos x}{6-(1-\sin^2 x)-4\sin x}dx = \int \frac{(4\sin x-1)\cos x}{\sin^2 x - 4\sin x + 5}dx$$

Put $\sin x = t$, so that $\cos x\, dx = dt$.

$\therefore \quad I = \int \frac{(4t-1)dt}{(t^2-4t+5)}$ ...... (i)

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$2\lambda = 4,\ -4\lambda + \mu = -1$ ...... (ii)

$\lambda = 2,\ \mu = 7$

$\therefore \quad I = \int \frac{2(2t-4)+7}{t^2-4t+5}dt$ {using (i) and (ii)}

$= 2\int \frac{2t-4}{t^2-4t+5}dt + 7\int \frac{dt}{t^2-4t+5} = 2\log|t^2-4t+5| + 7\int \frac{dt}{t^2-4t+4-4+5}$

$= 2\log|t^2-4t+5| + 7\int \frac{dt}{(t-2)^2+(1)^2} = 2\log|t^2 - 4t + 5| + 7 \,.\, \tan^{-1}(t-2) + c$

$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + c.$ **Ans.**

Illustration 30 : The value of $\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$, is equal to -

(A) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(B) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(C) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(D) none of these

Solution : Here, $I = \int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$

Put $x = 3\cos 2\theta \Rightarrow dx = -6\sin 2\theta d\theta$

$= \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)d\theta$

$= \int \frac{\sin\theta}{\cos\theta}.\sin^{-1}(\sin\theta).(-6\sin 2\theta)d\theta = -6\int \theta.(2\sin^2\theta)d\theta$

$= -6\int \theta(1-\cos 2\theta)d\theta = -6\left\{\frac{\theta^2}{2} - \int \theta\cos 2\theta\, d\theta\right\}$

$= -6\left\{\frac{\theta^2}{2} - \left(\theta\frac{\sin 2\theta}{2} - \int 1.\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\} = -3\theta^2 + 6\left\{\theta\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + c$

$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$ **Ans. (A)**

Illustration 31 : Evaluate : $\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}dx$

Solution : $I = \int \dfrac{\tan\left(\dfrac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx = \int \dfrac{(1-\tan^2 x)dx}{(1+\tan x)^2\cos^2 x\sqrt{\tan^3 x+\tan^2 x+\tan x}}$

$$I = \int \frac{-\left(1-\dfrac{1}{\tan^2 x}\right)\sec^2 x dx}{\left(\tan x+2+\dfrac{1}{\tan x}\right)\sqrt{\tan x+1+\dfrac{1}{\tan x}}}$$

let, $y = \sqrt{\tan x+1+\dfrac{1}{\tan x}} \Rightarrow 2y\ \ dy = \left(\sec^2 x - \dfrac{1}{\tan^2 x}.\sec^2 x\right)dx$

$\therefore\quad I = \int \dfrac{-2ydy}{(y^2+1).y} = -2\int \dfrac{dy}{1+y^2}$

$= -\ 2\tan^{-1} y + c = -2\tan^{-1}\left(\sqrt{\tan x+1+\dfrac{1}{\tan x}}\right)+c$ Ans.

ANSWERS FOR DO YOURSELF

1 : (i) $\dfrac{1}{36}\tan^{-1}\left(\dfrac{4x^3}{3}\right)+c$ (ii) $\sin x-\dfrac{1}{3}\sin^3 x+c$

2 : (i) $\sqrt{\dfrac{3-x}{x-2}}-\sin^{-1}\sqrt{3-x}+c$ (ii) $\ell n\left[\left(x+\dfrac{1}{2}\right)+\sqrt{x^2+x+1}\right]+c$

3 : (i) $xe^x - e^x + c$ (ii) $-\dfrac{1}{2}x^2\cos(x^2)+\sin(x^2)+c$

4 : (i) $e^x\ \tan^{-1} + c$ (ii) $\dfrac{1}{2}e^{x^2}\sin(x^2)+c$

5 : (i) $x\ \tan(e^x) + c$ (ii) $x\ell nx + c$

6 : (i) $-\dfrac{1}{2}\ell n|x+1|+\dfrac{7}{2}\ell n|x+3|+c$ (ii) $\ell n|x+2|+\dfrac{3}{x+2}+c$

7 : (i) $\dfrac{\sqrt{3}}{2}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$ (ii) $5\sqrt{x^2+4x+1}-6\ell n\left[(x+2)+\sqrt{x^2+4x+1}\right]+c$

8 : (i) $\tan^{-1}\left(\dfrac{x^2-1}{x}\right)+c$ (ii) $\dfrac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{2}x}\right)-\dfrac{1}{2}\ell n\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right]$

9 : (i) $-\dfrac{1}{2}\ell n\left(\dfrac{x^2+1}{x^2}\right)+c$ (ii) $-\left(1+\dfrac{1}{x^3}\right)^{1/3}+c$ (iii) $-\dfrac{1}{2}\left(1+\dfrac{1}{x^3}\right)^{2/3}+c$

10 : (i) $\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\tan x\right)+c$ (ii) $\dfrac{2}{\sqrt{15}}\tan^{-1}\left(\dfrac{8\tan x+1}{\sqrt{15}}\right)+c$

11 : (i) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{3\tan x/2+1}{2\sqrt{2}}\right)+c$ (ii) $\dfrac{1}{2\sqrt{6}}\ell n\left|\dfrac{\sqrt{6}+\tan x/2-2}{\sqrt{6}-\tan x/2+2}\right|+c$

12 : (i) $\dfrac{1}{2}x-\dfrac{1}{2}\ell n\left|\sin x+\cos x\right|+c$ (ii) $\dfrac{12}{13}x-\dfrac{5}{13}\ell n\left|3\cos x+2\sin x\right|+C$

13 : (i) $\dfrac{1}{3}\tan^3 x+c$ (ii) $\dfrac{2}{3}\tan^{3/2}x+c$ (iii) $\dfrac{1}{3}\sin^3 x-\dfrac{2}{5}\sin^5 x+\dfrac{1}{7}\sin^7 x+c$.

14 : (i) $2\sqrt{x+1}+\dfrac{3}{2}\ell n\left|\dfrac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$ (ii) $-\dfrac{1}{2}\sqrt{1+x^2}+c$

EXERCISE - 01

CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If $f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx$, $x \neq 0$ then $\lim_{x\to 0} f'(x)$ is equal to-

(A) 0 (B) 1 (C) 2 (D) 1/2

2. $\int 4\sin x \cos\frac{x}{2}\cos\frac{3x}{2} dx$ is equal to -

(A) $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + c$ (B) $\cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + c$

(C) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + c$ (D) $\cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + c$

3. $\int \frac{8x+13}{\sqrt{4x+7}} dx$ is equal to -

(A) $\frac{1}{6}(8x + 11)\sqrt{4x+7} + c$ (B) $\frac{1}{6}(8x + 13)\sqrt{4x+7} + c$

(C) $\frac{1}{6}(8x + 9)\sqrt{4x+7} + c$ (D) $\frac{1}{6}(8x + 15)\sqrt{4x+7} + c$

4. $\int \left(\frac{\cos^8 x - \sin^8 x}{1 - 2\sin^2 x\cos^2 x}\right) dx$ equals -

(A) $-\frac{\sin 2x}{2} + c$ (B) $\frac{\sin 2x}{2} + c$ (C) $\frac{\cos 2x}{2} + c$ (D) $-\frac{\cos 2x}{2} + c$

5. Primitive of $\sqrt[3]{\frac{x}{(x^4-1)^4}}$ w.r.t. x is -

(A) $\frac{3}{4}\left(1+\frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$ (B) $-\frac{3}{4}\left(1+\frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$ (C) $\frac{4}{3}\left(1+\frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$ (D) $-\frac{4}{3}\left(1+\frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$

6. $\int (1 + 2x + 3x^2 + 4x^3 + \ldots\ldots)\, dx$ $(|x| < 1)$ -

(A) $(1 + x)^{-1} + c$ (B) $(1 - x)^{-1} + c$ (C) $(1 + x)^{-2} + c$ (D) none of these

7. $\int \frac{x\ dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$ is equal to -

(A) $\frac{1}{2}\ln\left(1+\sqrt{1+x^2}\right) + c$ (B) $2\sqrt{1+\sqrt{1+x^2}} + c$

(C) $2\left(1+\sqrt{1+x^2}\right) + c$ (D) none of these

8. $\int \frac{\ell n|x|}{x\sqrt{1+\ell n|x|}} dx$ equals -

(A) $\frac{2}{3}\sqrt{1+\ell n|x|}(\ell n|x|-2) + c$ (B) $\frac{2}{3}\sqrt{1+\ell n|x|}(\ell n|x|+2) + c$

(C) $\frac{1}{3}\sqrt{1+\ell n|x|}(\ell n|x|-2) + c$ (D) $2\sqrt{1+\ell n|x|}(3\ell n|x|-2) + c$

9. If $\int \frac{x^4+1}{x(x^2+1)^2}dx = A\ \ell n|x| + \frac{B}{1+x^2}$ + c, where c is the constant of integration then :

(A) A = 1; B = –1 (B) A = –1; B = 1 (C) A = 1; B = 1 (D) A = –1 ; B = –1

10. $\int \left(\frac{x}{1+x^5}\right)^{3/2} dx$ equals -

(A) $\frac{2}{5}\sqrt{\frac{x^5}{1+x^5}}+c$ (B) $\frac{2}{5}\sqrt{\frac{x}{1+x^5}}+c$ (C) $\frac{2}{5}\frac{1}{\sqrt{1+x^5}}+c$ (D) none of these

11. $\int \sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x\ dx$ equals -

(A) $\frac{\sin 16x}{1024}+c$ (B) $-\frac{\cos 32x}{1024}+c$ (C) $\frac{\cos 32x}{1096}+c$ (D) $-\frac{\cos 32x}{1096}+c$

12. Identify the correct expression

(A) $x\int \ell nx\,dx = x^2\ell n|x| - x^2 + c$ (B) $x\int \ell n|x|dx = xe^x + c$

(C) $x\int e^x dx = xe^x + c\,x$ (D) $\int \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

13. $\int x.\frac{\ell n\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals -

(A) $\sqrt{1+x^2}\ \ell n\left(x+\sqrt{1+x^2}\right)-x+c$ (B) $\frac{x}{2}.\ \ell n^2\left(x+\sqrt{1+x^2}\right)-\frac{x}{\sqrt{1+x^2}}+c$

(C) $\frac{x}{2}.\ell n^2\left(x+\sqrt{1+x^2}\right)+\frac{x}{\sqrt{1+x^2}}+c$ (D) $\sqrt{1+x^2}\ \ell n\left(x+\sqrt{1+x^2}\right)+x+c$

14. If $\int \frac{dx}{(x+2)(x^2+1)} = a\ell n(1+x^2) + b\tan^{-1}x + \frac{1}{5}\ \ell n|x+2| + C$ then-

(A) $a = -\frac{1}{10}$, $b = -\frac{2}{5}$ (B) $a = \frac{1}{10}$, $b = -\frac{2}{5}$ (C) $a = -\frac{1}{10}$, $b = \frac{2}{5}$ (D) $a = \frac{1}{10}$, $b = \frac{2}{5}$

15. $\int \frac{(x-1)^2}{x^4+2x^2+1}dx$ equals -

(A) $\frac{x^3}{3}+x+\frac{x}{x^2+1}+c$ (B) $\frac{x^5+x^3+x+3}{3(x^2+1)}+c$ (C) $\frac{x^5+4x^3+3x+3}{3(x^2+1)}+c$ (D) None of these

16. $\int \frac{x^2-4}{x^4+24x^2+16}dx$ equals -

(A) $\frac{1}{4}\tan^{-1}\left(\frac{(x^2+4)}{4x}\right)+c$ (B) $-\frac{1}{4}\cot^{-1}\left(\frac{(x^2+4)}{x}\right)+c$

(C) $-\frac{1}{4}\cot^{-1}\left(\frac{4(x^2+4)}{x}\right)+c$ (D) $\frac{1}{4}\cot^{-1}\left(\frac{(x^2+4)}{x}\right)+c$

17. $\int \frac{x^4-4}{x^2\sqrt{4+x^2+x^4}}dx$ equals-

(A) $\frac{\sqrt{4+x^2+x^4}}{x}+c$ (B) $\sqrt{4+x^2+x^4}+c$ (C) $\frac{\sqrt{4+x^2+x^4}}{2}+c$ (D) $\frac{\sqrt{4+x^2+x^4}}{2\ x}+c$

18. $\int \frac{x^9}{(x^2+4)^6}dx$ is equal to -

(A) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+c$ (B) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+c$

(C) $\frac{1}{10x}(1+4x^2)^{-5}+c$ (D) $\frac{1}{40}(1+4x^{-2})^{-5}+c$

19. If $\int \frac{dx}{5+4\cos x} = a\tan^{-1}\left(b\tan\frac{x}{2}\right)+c$, then-

(A) $a=\frac{2}{3},\ b=-\frac{1}{3}$ (B) $a=\frac{2}{3},\ b=\frac{1}{3}$

(C) $a=-\frac{2}{3},\ b=\frac{1}{3}$ (D) $a=-\frac{2}{3},\ b=-\frac{1}{3}$

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

20. Primitive of $\sqrt{1+2\tan x(\sec x+\tan x)}$ w.r.t.x is -

(A) $\ell n|\sec x|-\ell n|\sec x-\tan x|+c$ (B) $\ell n|\sec x+\tan x|+\ell n|\sec x|+c$

(C) $2\ell n\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right|+c$ (D) $\ell n|1+\tan x(\sec x+\tan x)|+c$

21. $\int \sin 2x\,dx$ equals -

(A) $-\frac{\cos 2x}{2}+c$ (B) $\frac{\sin^2 x}{2}+c$ (C) $-\frac{\cos^2 x}{2}+c$ (D) $\frac{\cos 2x}{2}+c$

22. $\int \frac{dx}{x^3\left(1-\frac{1}{2x^2}\right)}$ equals-

(A) $\ell n|2x^2-1|+2\ell n|x|+c$ (B) $\ell n|2x^2-1|-2\ell n|x|+c$

(C) $\ell n|2x^2-1|-\ell n(x^2)-\ell n2+c$ (D) $\ell n\left|1-\frac{1}{2x^2}\right|+c$

23. If $\int e^{3x}\cos 4x\,dx = e^{3x}(A\sin 4x+B\cos 4x)+c$, then -

(A) 4A = 3B (B) 2A = 3B (C) 3A = 4B (D) 4A + 3B = 1

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	B	A	B	B	B	B	A	C	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	A	C	D	A	A	D	B	A,B,D
Que.	21	22	23							
Ans.	A,B,C	B,C,D	C,D							

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{\sqrt{2}(\cos x+\sin x)}dx$ equals -

(A) $\sec^{-1}(\sin x+\cos x)+c$
(B) $\sec^{-1}(\sin x-\cos x)+c$
(C) $\ell n\left|(\sin x+\cos x)+\sqrt{\sin 2x}\right|+c$
(D) $\ell n\left|(\sin x-\cos x)+\sqrt{\sin 2x}\right|+c$

2. $\int \frac{\sin x+4\sin 3x+6\sin 5x+3\sin 7x}{\sin 2x+3\sin 4x+3\sin 6x}dx$ equals -

(A) $-2\sin x+c$ (B) $2\sin x+c$ (C) $-2\cos x+c$ (D) $2\cos x+c$

3. $\int \frac{1-x^7}{x(1+x^7)}dx$ equals -

(A) $\ell n|x|+\frac{2}{7}\ell n\left|1+x^7\right|+c$
(B) $\ell n|x|-\frac{2}{4}\ell n\left|1-x^7\right|+c$
(C) $\ell n|x|-\frac{2}{7}\ell n\left|1+x^7\right|+c$
(D) $\ell n|x|+\frac{2}{4}\ell n\left|1-x^7\right|+c$

4. $\int \frac{x^3dx}{\sqrt{1+x^2}}$ is equal to -

(A) $\frac{1}{3}\sqrt{1+x^2}\,(2+x^2)+c$
(B) $\frac{1}{3}\sqrt{1+x^2}\,(x^2-1)+c$
(C) $\frac{1}{3}(1+x^2)^{3/2}+c$
(D) $\frac{1}{3}\sqrt{1+x^2}\,(x^2-2)+c$

5. $\int \sin^2(\ell n\, x)\, dx$ is equal to -

(A) $\frac{x}{10}(5+2\sin(2\ell nx)+\cos(2\ell nx))+c$
(B) $\frac{x}{10}(5+2\sin(2\ell nx)-\cos(2\ell nx))+c$
(C) $\frac{x}{10}(5-2\sin(2\ell nx)-\cos(2\ell nx))+c$
(D) $\frac{x}{10}(5-2\sin(2\ell nx)+\cos(2\ell nx))+c$

6. $\int \frac{x^2+\cos^2 x}{1+x^2}.\text{cosec}^2x\, dx$ is equal to -

(A) $\cot x+\tan^{-1}x+c$ (B) $\cot x-\tan^{-1}x+c$ (C) $-\cot x-\tan^{-1}x+c$ (D) $\tan^{-1}x-\cot x+c$

7. $\int e^x\left(\frac{x^2-3}{(x+3)^2}\right)dx$, equals-

(A) $e^x.\frac{x}{x+3}+c$ (B) $e^x\left(2-\frac{6}{x+3}\right)+c$ (C) $e^x\left(1-\frac{6}{x+3}\right)+c$ (D) $e^x.\frac{3}{x+3}+c$

8. $\int e^{\tan^{-1}x}(1+x+x^2).d(\cot^{-1}x)$ is equal to -

(A) $-e^{\tan^{-1}x}+c$ (B) $e^{\tan^{-1}x}+c$ (C) $-x.e^{\tan^{-1}x}+c$ (D) $x.e^{\tan^{-1}x}+c$

9. $\int e^x\frac{(1+n.x^{n-1}-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}}dx$ is equal to -

(A) $e^x\sqrt{\frac{1-x^n}{1+x^n}}+c$ (B) $e^x\sqrt{\frac{1+x^n}{1-x^n}}+c$ (C) $-e^x\sqrt{\frac{1-x^n}{1+x^n}}+c$ (D) $-e^x\sqrt{\frac{1+x^n}{1-x^n}}+c$

10. $\int e^{x^4}(x+x^3+2x^5)e^{x^2}dx$ is equal to -

(A) $\frac{1}{2}xe^{x^2}.e^{x^4}+c$ (B) $\frac{1}{2}x^2e^{x^4}+c$ (C) $\frac{1}{2}e^{x^2}.e^{x^4}+c$ (D) $\frac{1}{2}x^2e^{x^2}.e^{x^4}+c$

11. Primitive of $\frac{3x^4-1}{\left(x^4+x+1\right)^2}$ w.r.t. x is -

(A) $\frac{x}{x^4+x+1}+c$ (B) $-\frac{x}{x^4+x+1}+c$ (C) $\frac{x+1}{x^4+x+1}+c$ (D) $-\frac{x+1}{x^4+x+1}+c$

12. $\int\frac{dx}{x^4\,[x\,(x^5-1)]^{1/3}}$ equals -

(A) $\frac{3}{2}\left\{\frac{x^5-1}{x^5}\right\}^{2/3}+c$ (B) $\frac{3}{10}\left\{\frac{x^5-1}{x^5}\right\}^{2/3}+c$ (C) $\frac{3}{4}\left\{\frac{x^5-1}{x^5}\right\}^{2/3}+c$ (D) $\frac{3}{5}\left\{\frac{x^5-1}{x^5}\right\}^{2/3}+c$

13. $\int\frac{\sin x}{\sin 4x}dx$ is equal to -

(A) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+\frac{1}{8}\ell n\left|\frac{1+\sin x}{1-\sin x}\right|+c$ (B) $\frac{1}{2\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|-\frac{1}{8}\ell n\left|\frac{1+\sin x}{1-\sin x}\right|+c$

(C) $\frac{1}{4\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+\frac{1}{8}\ell n\left|\frac{1+\sin x}{1-\sin x}\right|+c$ (D) $\frac{1}{4\sqrt{2}}\ell n\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|-\frac{1}{8}\ell n\left|\frac{1+\sin x}{1-\sin x}\right|+c$

14. The value of integral $\int\frac{d\theta}{\cos^3\theta\sqrt{\sin 2\theta}}$ can be expressed as irrational function of $\tan\theta$ as -

(A) $\frac{\sqrt{2}}{5}\left(\sqrt{\tan^2\theta+5}\right)\tan\theta+c$ (B) $\frac{2}{5}\left(\tan^2\theta+5\right)\sqrt{\tan\theta}+c$

(C) $\frac{\sqrt{2}}{5}\left(\tan^2\theta+5\right)\sqrt{\tan\theta}+c$ (D) $\sqrt{\frac{2}{5}}\left(\tan^2\theta+5\right)\sqrt{\tan\theta}+c$

15. If $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx = ax + b\ell n[2\sin x + 3\cos x| + c$, then -

(A) $a=-\frac{12}{13}$, $b=\frac{15}{39}$ (B) $a=\frac{17}{13}$, $b=\frac{6}{13}$ (C) $a=\frac{12}{13}$, $b=-\frac{15}{39}$ (D) $a=-\frac{17}{13}$, $b=-\frac{1}{192}$

16. $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx$ is equal to -

(A) $\ell n\left|x-\sqrt{x^2-1}\right| - \tan^{-1}x + c$

(B) $\ell n\left|x+\sqrt{x^2-1}\right| - \tan^{-1}x + c$

(C) $\ell n\left|x-\sqrt{x^2-1}\right| - \sec^{-1}x + c$

(D) $\ell n\left|x+\sqrt{x^2-1}\right| - \sec^{-1}x + c$

17. $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to -

(A) $\frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + c$ (B) $\frac{2(1+\sqrt{x})}{\sqrt{1-x}} + c$ (C) $\frac{2(\sqrt{x}-1)}{\sqrt{x-1}} + c$ (D) $\frac{2(1+\sqrt{x})}{\sqrt{x-1}} + c$

18. Let $f'(x) = 3x^2.\sin\frac{1}{x} - x\cos\frac{1}{x}$, $x \neq 0$, $f(0) = 0$, $f\left(\frac{1}{\pi}\right) = 0$, then which of the following is/are not correct.

(A) f(x) is continuous at x = 0

(B) f(x) is non-differentiable at x = 0

(C) f'(x) is discontinuous at x = 0

(D) f'(x) is differentiable at x = 0

19. $\int \frac{1}{x^2-1}\, \ell n\, \frac{x-1}{x+1}\, dx$ equals -

(A) $\frac{1}{2}\ell n^2\left|\frac{x-1}{x+1}\right|+c$ (B) $\frac{1}{4}\ell n^2\left|\frac{x-1}{x+1}\right|+c$ (C) $\frac{1}{2}\ell n^2\left|\frac{x+1}{x-1}\right|+c$ (D) $\frac{1}{4}\ell n^2\left|\frac{x+1}{x-1}\right|+c$

20. $\int \frac{dx}{\sqrt{x-x^2}}$ equals, where $x \in \left(\frac{1}{2},1\right)$ -

(A) $2\sin^{-1}\sqrt{x}+c$

(B) $\sin^{-1}(2x-1)+c$

(C) $c-\cos^{-1}(2x-1)$

(D) $\cos^{-1} 2\sqrt{x-x^2}+c$

21. $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}dx$ is equal to -

(A) $\cot^{-1}\left(\cot^2 x\right)+c$ (B) $-\cot^{-1}\left(\tan^2 x\right)+c$ (C) $\tan^{-1}\left(\tan^2 x\right)+c$ (D) $-\tan^{-1}\left(\cos 2x\right)+c$

BRAIN TEASERS **ANSWER KEY** EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	B	C	D	C	C	C	C	B	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	D	C	C	D	A	B,C,D	B,D	A,B,C,D
Que.	21									
Ans.	A,B,C,D									

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

FILL IN THE BLANKS

1. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\,dx = Ax + B\log(9e^{2x} - 4) + C$, then A = ……., B = ……. and C = ……

2. If the graph of the antiderivative F(x) of $f(x) = \log(\log x) + (\log x)^{-2}$ passes through (e, 1998 − e) then the term independent of x in F(x) is …….

3. Let F(x) be the antiderivative of f(x) = 3cosx − 2sinx whose graph passes through the point (π/2, 1). Then F(π/2) = …….

4. Let f be a function satisfying $f''(x) = x^{-3/2}$, f'(4) = 2 and f(0) = 0. Then f(784) is equal to ……..

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1. The antiderivative of

	Column-I		Column-II
(A)	$f(x) = \frac{1}{(a^2+b^2)-(a^2-b^2)\cos x}$ is	(p)	$\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan\frac{x}{2}\right) + c$
(B)	$f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$ is	(q)	$\frac{1}{a^2\sin\alpha}\tan^{-1}\left(\frac{\tan x}{\sin\alpha}\right) + c,\ \alpha = \cos^{-1}\frac{b}{a}$
(C)	$f(x) = \frac{1}{a\cos x + b\sin x}$ is	(r)	$\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan x\right) + c$
(D)	$f(x) = \frac{1}{a^2 - b^2\cos^2 x}$ is ; $(a^2 > b^2)$	(s)	$\frac{1}{\sqrt{a^2+b^2}}\log\left\|\tan\frac{1}{2}\left(x + \tan^{-1}\frac{a}{b}\right)\right\| + c$

2. $\int f(x)\,dx$ when

	Column-I		Column-II
(A)	$f(x) = \frac{1}{(a^2+x^2)^{3/2}}$	(p)	$c - \frac{1}{a}\sin^{-1}\frac{a}{\|x\|}$
(B)	$f(x) = \frac{x^2}{\sqrt{a^2-x^2}}$	(q)	$\frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + c$
(C)	$f(x) = \frac{1}{(x^2-a^2)^{3/2}}$	(r)	$c - \frac{x}{a^2\sqrt{x^2-a^2}}$
(D)	$f(x) = \frac{1}{x\sqrt{x^2-a^2}}$	(s)	$\frac{x}{a^2\sqrt{x^2+a^2}} + c$

ASSERTION & REASON

In each of the following questions, a statement of Assertion (A) is given followed by a corresponding statement of Reason (R) just below it . Of the statements mark the correct answer as

(A) Statement-I is True, Statement-II is True ; Statement-II is a correct explantion for Statement-I

(B) Statement-I is True, Statement-II is True ; Statement-II is NOT a correct explantion for Statement-I

(C) Statement-I is True, Statement-II is False.

(D) Statement-I is False, Statement-II is True.

1. If $D(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, where f_1, f_2, f_3 are differentiable function and a_2, b_2, c_2, a_3, b_3, c_3 are constants.

Statement - I : $\int D(x)\,dx = \begin{vmatrix} \int f_1(x)dx & \int f_2(x)dx & \int f_3(x)dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + c$

Because

Statement - II : Integration of sum of several function is equal to sum of integration of individual functions.

(A) A (B) B (C) C (D) D

2. **Statement - I :** If $a > 0$ and $b^2 - 4ac < 0$, then the values of integral $\int \frac{dx}{ax^2+bx+c}$ will be of the type $\mu \tan^{-1}\frac{x+A}{B}$ + c. where A, B, C, μ are constants.

Because

Statement - II : If $a > 0$, $b^2 - 4ac < 0$, then $ax^2 + bx + c$ can be written as sum of two squares.

(A) A (B) B (C) C (D) D

3. If y is a function of x such that $y(x - y)^2 = x$.

Statement - I : $\int \frac{dx}{x-3y} = \frac{1}{2}\log[(x - y)^2 - 1]$

Because

Statement - II : $\int \frac{dx}{x-3y} = \log(x - 3y) + c$.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

In calculating a number of integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly and in a more concise form by using the so-called generalized formula for integration by parts

$$\int u(x)\,v(x)dx = u(x)\,v_1(x) - u'(x)v_2(x) + u''(x)\,v_3(x) - \ldots\ldots + (-1)^{n-1}u^{n-1}(x)\,v_n(x) - (-1)^{n-1}\int u^n(x)\,v_n(x)\,dx$$

where $v_1(x) = \int v(x)dx$, $v_2(x) = \int v_1(x)\,dx$ …., $v_n(x) = \int v_{n-1}(x)\,dx$

Of course, we assume that all derivatives and integrals appearing in this formula exist. The use of the generalized formula for integration by parts is especially useful when calculating $\int P_n(x)\,Q(x)\,dx$, where $P_n(x)$, is polynomial of degree n and the factor Q(x) is such that it can be integrated successively n + 1 times.

1. If $\int (x^3 - 2x^2 + 3x - 1)\cos 2x\,dx = \frac{\sin 2x}{4}u(x) + \frac{\cos 2x}{8}v(x) + c$, then -

(A) $u(x) = x^3 - 4x^2 + 3x$
(B) $u(x) = 2x^3 - 4x^2 + 3x$
(C) $v(x) = 3x^2 - 4x + 3$
(D) $v(x) = 6x^2 - 8x$

2. If $\int e^{2x}.x^4 dx = \frac{e^{2x}}{2}f(x) + C$ then f(x) is equal to -

(A) $\left(x^4 - 2x^3 + 3x^2 - 3x + \frac{3}{2}\right)\frac{1}{2}$
(B) $x^4 - x^3 + 2x^2 - 3x + 2$
(C) $x^4 - 2x^3 + 3x^2 - 3x + \frac{3}{2}$
(D) $x^4 - 2x^3 + 2x^2 - 3x + \frac{3}{2}$

Comprehension # 2

Integrals of class of functions following a definite pattern can be found by the method of reduction and recursion. Reduction formulas make it possible to reduce an integral dependent on the index n > 0, called the order of the integral, to an integral of the same type with a smaller index. Integration by parts helps us to derive reduction formulas. (Add a constant in each question)

1. If $I_n = \int \frac{dx}{(x^2+a^2)^n}$ then $I_{n+1} + \frac{1-2n}{2n}\cdot\frac{1}{a^2}\ I_n$ is equal to -

(A) $\frac{x}{(x^2+a^2)^n}$ (B) $\frac{1}{2na^2}\ \frac{1}{(x^2+a^2)^{n-1}}$ (C) $\frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n}$ (D) $\frac{1}{2na^2}\cdot\frac{1}{(x^2+a^2)}$

2. If $I_{n,\,-m} = \int \frac{\sin^n x}{\cos^m x}dx$ then $I_{n,\,-m} + \frac{n-1}{m-1} I_{n-2,\ 2-m}$ is equal to-

(A) $\frac{\sin^{n-1} x}{\cos^{m-1} x}$ (B) $\frac{1}{(m-1)}\ \frac{\sin^{n-1} x}{\cos^{m-1} x}$ (C) $\frac{1}{(n-1)}\ \frac{\sin^{n-1} x}{\cos^{m-1} x}$ (D) $\frac{n-1}{m-1}\ \frac{\sin^{n-1} x}{\cos^{m-1} x}$

3. If $u_n = \int \frac{x^n}{\sqrt{ax^2+2bx+c}}dx$, then $(n + 1)au_{n+1} + (2n + 1)bu_n + ncu_{n-1}$ is equal to -

(A) $x^{n-1}\sqrt{ax^2+2bx+c}$ (B) $\frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}$ (C) $\frac{x^n}{\sqrt{ax^2+2bx+c}}$ (D) $x^n\sqrt{ax^2+2bx+c}$

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE-3

- ***Fill in the Blanks***

1. $\frac{-3}{2}, \frac{35}{36}$, any real value **2.** 1998 **3.** 1 **4.** 2240

- ***Match the Column***

1. (A) → p; (B) → r; (C) → s; (D) → q **2.** (A) → s; (B) → q,; (C) → r; (D) → p

- ***Assertion & Reason***

1. (A) **2.** (A) **3.** (C)

- ***Comprehension Based Questions***

Comprehension # 1 : **1.** (B) **2.** (C)

Comprehension # 2 : **1.** (C) **2.** (B) **3.** (D)

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

Evaluate the following Indefinite integrals :

1. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

2. $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

3. $\int \tan x.\tan 2x.\tan 3x\ dx$

4. $\int \left[\frac{\sqrt{x^2+1}\ \left[\ell n(x^2+1)-2\ \ell n x\right]}{x^4}\right]dx$

5. Integrate $\frac{1}{2}f'(x)$ w.r.t. x^4, where $f(x) = \tan^{-1}x + \ell n\sqrt{1+x} - \ell n\sqrt{1-x}$

6. $\int \sqrt{\frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x + \cot x}}.\frac{\sec x}{\sqrt{1+2\sec x}}dx$

7. $\int \frac{(ax^2-b)\ dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

8. $\int \frac{x^2}{(x\sin x+\cos x)^2}dx$

9. $\int \cos 2\theta.\ell n\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\ d\theta$

10. $\int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right]\ell n x\,dx$

11. $\int \frac{x\ell n x}{(x^2-1)^{3/2}}\ dx$

12. $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$ [JEE 99]

13. $\int \frac{3x^2+1}{(x^2-1)^3}\ dx$

14. $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ [JEE 84]

15. $\int \frac{dx}{\sin^2 x+\sin 2x}$

16. $\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx$

17. $\int \frac{\cos x-\sin x}{7-9\sin 2x}dx$

18. $\int \frac{\cos^2 x}{1+\tan x}dx$

19. $\int \left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ [JEE 89]

20. $\int \frac{(\cos 2x)^{1/2}}{\sin x}\ dx$ [JEE 87]

21. $\int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx$ [JEE 92]

CONCEPTUAL SUBJECTIVE EXERCISE | ANSWER KEY | EXERCISE-4(A)

1. $\operatorname{cosec}(b-a).\ell n\left|\frac{\sin(x-b)}{\sin\ (x-a)}\right|+c$

2. $-\frac{x+1}{x^5+x+1}+c$

3. $\left[-\ell n(\sec x)-\frac{1}{2}\ell n(\sec 2x)+\frac{1}{3}\ell n(\sec 3x)\right]+c$

4. $\frac{(x^2+1)\ \sqrt{x^2+1}}{9x^3}.\left[2-3\ \ell n\left(1+\frac{1}{x^2}\right)\right]$

5. $-\ell n\left|1-x^4\right|+c$

6. $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)+c$

7. $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$

8. $\frac{\sin x - x\cos x}{x\sin x+\cos x}+c$

9. $\frac{1}{2}(\sin 2\theta)\ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\ell n(\sec 2\theta) + c$

10. $\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+c$

11. arc secx $-\frac{\ell n x}{\sqrt{x^2-1}}+c$

12. $\frac{3}{2}\tan^{-1}x-\frac{1}{2}\ell n(1+x)+\frac{1}{4}\ell n(1+x^2)+\frac{x}{1+x^2}+c$

13. $C-\frac{x}{(x^2-1)^2}$

14. $-\left(1+\frac{1}{x^4}\right)^{1/4}+c$

15. $\frac{1}{2}\ell n\left|\frac{\tan x}{\tan x+2}\right|+c$

16. $-\frac{3(1+4\tan^2 x)}{8\ (\tan x)^{8/3}}+c$

17. $\frac{1}{24}\ell n\frac{(4+3\ \sin x\ +3\ \cos x)}{(4-3\ \sin x-3\ \cos x)}+c$

18. $\frac{1}{4}\ell n(\cos + \sin x) + \frac{x}{2} + \frac{1}{8}(\sin 2x + \cos 2x)$

19. $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

20. $\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right] - \log(\cot x + \sqrt{\cot^2 x-1}) + c$

21. $\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/3}+6x^{1/6}-12x^{1/12}+12\log|x^{1/12}+1|-4x^{1/4}+c$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. $\int \frac{\cos 8x - \cos 7x}{1+2\cos 5x} dx$ 2. $\int \sqrt{x+\sqrt{x^2+2}}\, dx$ 3. $\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}} dx$ 4. $\int \frac{\cot x\, dx}{(1-\sin x)(\sec x+1)}$

5. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ 6. $\int \frac{dx}{\sec x + \operatorname{cosec} x} dx$ 7. $\int \frac{dx}{\sin x\sqrt{\sin(2x+\alpha)}}$

8. $\int \frac{\ell n\left(\cos x + \sqrt{\cos 2x}\right)}{\sin^2 x} dx$ 9. $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}} dx$

10. Let $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 2\alpha x + \beta x^2 \\ 5x + \gamma x^2 + 3 \end{bmatrix}$, $\forall$ x $\in$ R and f(x) is a differentiable function satisfying,

$f(xy) = f(x) + x^2 (y^2 - 1) + x (y - 1)$; $\forall$ x, y $\in$ R and f(1) = 3 . Evaluate $\int \frac{\alpha x^2 + \beta x + \gamma}{f(x)} dx$

11. $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{1+3\sin 2x} dx$ 12. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

13. Let f(x) is a quadratic function such that f(0) = 1 and $\int \frac{f(x)dx}{x^2(x+1)^3}$ is a rational function, find the value of f'(0)

14. $\int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x}\, dx$ 15. $\int \frac{x}{(7x-10-x^2)^{3/2}}\, dx$

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)

1. $\frac{1}{6}(2\sin 3x - 3\sin 2x) + c$ 2. $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - \frac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}} + c$

3. $\cos a.\text{arc}\cos\left(\frac{\cos x}{\cos a}\right) - \sin a.\ell n\left(\sin x + \sqrt{\sin^2 x - \sin^2 a}\right) + c$ 4. $\frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + c$

5. $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} + \text{arc}\cos\sqrt{x} + c$ 6. $\frac{1}{2}\left[\sin x - \cos x - \frac{1}{\sqrt{2}}\ell n\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right] + c$

7. $-\frac{1}{\sqrt{\sin\alpha}}\ell n\left[\cot x + \cot\alpha + \sqrt{\cot^2 x + 2\cot\alpha\cot x - 1}\right] + c$

8. $\frac{\sqrt{\cos 2x}}{\sin x} - x - \cot x.\ell n\left(e\left(\cos x + \sqrt{\cos 2x}\right)\right) + c$ 9. $e^x\sqrt{\frac{1+x}{1-x}} + c$

10. $3x - \ell n\left(\sqrt{x^2+x+1}\right) + \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$ 11. $\tan^{-1}\left(\frac{\sqrt{2}\sin 2x}{\sin x + \cos x}\right) + c$

12. $(a + x)\ \text{arc}\tan\sqrt{\frac{x}{a}} - \sqrt{ax} + c$ 13. 3

14. $C - e^{\cos x}(x + \operatorname{cosec} x)$ 15. $\frac{2(7x-20)}{9\sqrt{7x-10-x^2}} + c$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$ [AIEEE-2002]

(1) $\tan x - x + C$ (2) $x + \tan x + C$ (3) $x - \tan x + C$ (4) $-x - \cot x + C$

2. $\int \frac{(\log x)}{x^2} dx$ [AIEEE-2002]

(1) $\frac{1}{2}(\log x + 1) + C$ (2) $-\frac{1}{x}(\log x + 1) + C$ (3) $\frac{1}{x}(\log x - 1) + C$ (4) $\log(x + 1) + C$

3. If $\int \frac{\sin x}{\sin(x-\alpha)} dx = Ax + B \log\sin(x - \alpha) + C$ then values of (A, B) is - [AIEEE-2004]

(1) $(\sin\alpha, \cos\alpha)$ (2) $(\cos\alpha, \sin\alpha)$ (3) $(-\sin\alpha, \cos\alpha)$ (4) $(-\cos\alpha, \sin\alpha)$

4. $\int \frac{dx}{\cos x - \sin x}$ is equal to- [AIEEE-2004]

(1) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$ (2) $\frac{1}{\sqrt{2}} \log\left|\cot\left(\frac{x}{2}\right)\right| + C$

(3) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + C$ (4) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$

5. $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is equals to - [AIEEE-2005]

(1) $\frac{\log x}{(\log x)^2 + 1} + C$ (2) $\frac{x}{x^2 + 1} + C$ (3) $\frac{xe^x}{1 + x^2} + C$ (4) $\frac{x}{(\log x)^2 + 1} + C$

6. $\int \frac{dx}{\cos x + \sqrt{3} \sin x}$ equals- [AIEEE-2007]

(1) $\frac{1}{2} \log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$ (2) $\frac{1}{2} \log\tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

(3) $\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$ (4) $\log\tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

7. The value of $\sqrt{2} \int \frac{\sin x\, dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is - [AIEEE-2008]

(1) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$ (2) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$ (3) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$ (4) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$

8. If the integral $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln|\sin x - 2 \cos x| + k$ then a is equal to : [AIEEE-2012]

(1) 2 (2) –1 (3) –2 (4) 1

9. If $\int f(x)dx = \Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to : [JEE (Main)-2013]

(1) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx\right] + C$

(2) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\Psi(x^3)dx + C$

(3) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$

(4) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^3\Psi(x^3)dx\right] + C$

PREVIOUS YEARS QUESTIONS ANSWER KEY EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9
Ans	3	2	2	4	4	1	3	1	3

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Evaluate : $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. [JEE 2001 (Mains) 5M out of 100]

2. For any natural number m, evaluate $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{\frac{1}{m}}dx$ where $x > 0$ [JEE 2002 5M out of 60]

3. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to - [JEE 2006, (3M, –1M) out of 184]

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$ (C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

4. Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \ge 2$ and $g(x) = \underbrace{(f\circ f\circ \ldots \circ f)}_{\text{f occurs n times}}(x)$. Then $\int x^{n-2}g(x)dx$ equals. [JEE 2007, 3M]

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$ (B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+K$ (C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$ (D) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+K$

5. Let F(x) be an indefinite integral of $\sin^2x$. [JEE 2007, 3M]

Statement-1 : The function F(x) satisfies $F(x+\pi) = F(x)$ for all real x.

because

Statement-2 : $\sin^2(x+\pi) = \sin^2x$ for all real x.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

6. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. [JEE 2008, 3M, –1M]

Then, for an arbitrary constant c, the value of J – I equals

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$ (B) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{2x}-e^{2x}+1}\right)+c$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$ (D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$

7. The integral $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ equals (for some arbitrary constant K) [JEE 2012, 3M, –1M]

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$ (B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$ (D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

PREVIOUS YEARS QUESTIONS — ANSWER KEY — EXERCISE-5 [B]

1. $(x+1)\tan^{-1}\frac{2(x+1)}{3}-\frac{3}{4}\ell n(4x^2+8x+13)+c$ 2. $\frac{\left(2x^{3m}+3x^{2m}+6x^m\right)^{\frac{m+1}{m}}}{6\ (m+1)}+C$

3. D 4. A 5. D 6. C 7. C